CONICAL SQUARE FUNCTIONS AND NON-TANGENTIAL MAXIMAL FUNCTIONS WITH RESPECT TO THE GAUSSIAN MEASURE

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ABSTRACT. We study, in $L^1(\mathbb{R}^n; \gamma)$ with respect to the gaussian measure, non-tangential maximal functions and conical square functions associated with the Ornstein-Uhlenbeck operator by developing a set of techniques which allow us, to some extent, to compensate for the non-doubling character of the gaussian measure. The main result asserts that conical square functions can be controlled in L^1-norm by non-tangential maximal functions. Along the way we prove a change of aperture result for the latter. This complements recent results on gaussian Hardy spaces due to Mauceri and Meda.

1. INTRODUCTION

Gaussian harmonic analysis, understood as the study of objects associated with the gaussian measure

$$d\gamma(x) = (2\pi)^{-n/2} \exp(-\tfrac{1}{2}|x|^2)\, dx$$

on $\mathbb{R}^n$, and the Ornstein-Uhlenbeck operator

$$Lf(x) = -\Delta f(x) + x \cdot \nabla f(x)$$

on function spaces such as $L^2(\mathbb{R}^n; \gamma)$, has recently gained new momentum following the development, by Mauceri and Meda [9], of an atomic Hardy space $H^1_{\mathrm{at}}(\mathbb{R}^n; \gamma)$, on which various functions of L give rise to bounded operators. Harmonic analysis in $L^p(\mathbb{R}^n; \gamma)$ has been relatively well established for some time, with results such as the boundedness of Riesz transforms going back to the work of Meyer and Pisier in the 1980's. The $p = 1$ case, however, has always proven to be difficult. Over the last 30 years, some weak type $(1, 1)$ estimates have been obtained, while others have been disproved (see the survey [13]). The proofs of these results rely on subtle decompositions and estimates of kernels. Until the seminal Mauceri-Meda paper appeared in 2007, a large part of euclidean harmonic analysis, such as end point estimates using Hardy and BMO spaces, seemed to have no gaussian counterpart. Gaussian harmonic analysis in $L^2(\mathbb{R}^n; \gamma)$ is relatively straightforward given the fact that the Ornstein-Uhlenbeck operator is diagonal with respect to the basis of Hermite polynomials. The $L^p(\mathbb{R}^n; \gamma)$ case, with $1 < p < \infty$, is harder but still manageable through kernel estimates. The end points $p = 1$ and $p = \infty$, however,

usually require techniques such as Whitney coverings and Calderón-Zygmund decompositions, for which the non-doubling nature of the gaussian measure, has, so far, not been overcome. Mauceri and Meda's paper [9], though, indicates a possible strategy. The authors used the notion of *admissible balls* which goes back to the work of Muckenhoupt [11]. These are balls $B(x, r)$ with the property that $r \leq a \min(1, \frac{1}{|x|})$ for some fixed admissibility parameter $a > 0$. On these admissible balls, the gaussian measure turns out to be doubling. The idea is then to follow classical arguments using admissible balls only. This is easier said than done. Indeed, admissible balls are very small when their centre is far away from the origin, whereas tools such as Whitney decompositions of open sets require the size of balls to be comparable to their distance to the boundary of the set, hence possibly very large. This may be why, although it contains many breakthrough results, Mauceri and Meda's paper [9] does not yet give a full theory of H^1 and BMO spaces for the gaussian measure. For instance, the boundedness of key operators such as maximal functions, conical square functions (area integrals), and above all Riesz transforms, is still missing. In fact, while this paper was in its final stages, Mauceri, Meda, and Sjögren [10] proved that Riesz transforms (more precisely *some* Riesz transforms, see their paper for the details) are bounded on the Mauceri-Meda Hardy space only in dimension one. This suggests that the 'correct' $H^1(\mathbb{R}^n; \gamma)$ space should be a modification of theirs.

In this paper, we take another step towards a satisfying $H^1(\mathbb{R}^n; \gamma)$ theory by studying, in $L^1(\mathbb{R}^n; \gamma)$, non-tangential square functions and maximal functions. These are gaussian analogues of the sublinear operators which, in the euclidean setting, are the cornerstones of the real variable theory of $H^1(\mathbb{R}^n)$. In the gaussian context, non-tangential maximal functions were first introduced in an unpublished work by Fabes and Forzani, who studied a gaussian counterpart of the Lusin area integral. Their L^p-boundedness was shown subsequently by Forzani, Scotto, and Urbina [6]. Our definition is an averaged version of a non-tangential maximal function from a subsequent paper of Pineda and Urbina [12]. The additional averaging adds some technical difficulties, but experience has shown (see e.g. [7]) that such averaging can be helpful in Hardy space theory and its applications (to boundary value problems, for instance).

Here we prove a change of aperture formula for the maximal function in the spirit of one of the key estimates of Coifman, Meyer and Stein [3]. We then show that the non-tangential square function is controlled by the non-tangential maximal function. Such estimates are central in Hardy space theory (see for instance [4, 5]). Thus, the purpose of this article is twofold. On the one hand, it contributes to the development of dyadic techniques in gaussian harmonic analysis, i.e., methods and results based on gaussian analogues of the decomposition of $\mathbb{R}^n$ into dyadic cubes, the related covering lemmas, and the corresponding H^1 and weak type $(1, 1)$ estimates. This makes the paper technical in nature, but we believe that the techniques developed here will find more applications, as gaussian harmonic analysis becomes more geometric and relies less on euclidean (after a change of variables) estimates of the Mehler kernel. On the other hand, this article gives some of the results required in the development of a gaussian Hardy space theory. When completed, such a theory will not only be satisfying from a pure harmonic analytic perspective, but it should also be applicable to stochastic partial differential equations (SPDE). Given the success of Hardy space techniques in deterministic PDE, one can think

that a gaussian analogue would similarly have applications to non-linear SPDE and stochastic boundary value problems.

Now let us state the main result of this paper. We set

$$m(x) := \min\left\{1, \frac{1}{|x|}\right\}$$

and let

$$\Gamma_x^a(\gamma) := \left\{(y,t) \in \mathbb{R}^n \times (0,\infty) \colon |y-x| < t < am(x)\right\}$$

denote the *admissible cone with parameter* $a > 0$ based at the point $x \in \mathbb{R}^n$. We denote by $(e^{-tL})_{t \geq 0}$ the Ornstein-Uhlenbeck semigroup acting on $L^p(\mathbb{R}^n; \gamma)$ for $1 < p < \infty$ (see the survey [13] and the references therein). For test functions $u \in C_c(\mathbb{R}^n)$ and admissibility parameter $a > 0$ we consider the *conical square function*

$$S_a u(x) = \left(\int_{\Gamma_x^a(\gamma)} \frac{1}{\gamma(B(y,t))} |t\nabla e^{-t^2 L} u(y)|^2 \, d\gamma(y) \, \frac{dt}{t}\right)^{\frac{1}{2}}.$$

and the *non-tangential maximal function*

$$T_a^* u(x) := \sup_{(y,t) \in \Gamma_x^a(\gamma)} \left(\frac{1}{\gamma(B(y,t))} \int_{B(y,t)} |e^{-t^2 L} u(z)|^2 d\gamma(z)\right)^{\frac{1}{2}}.$$

The main result of this paper reads as follows:

Theorem 1.1. *For each $a > 0$ there exists an $a' > 0$ such that the conical square function S_a is controlled by the non-tangential maximal function $T_{a'}^*$, in the sense that*

$$\|S_a u\|_{L^1(\mathbb{R}^n; \gamma)} \lesssim \|T_{a'}^* u\|_{L^1(\mathbb{R}^n; \gamma)}$$

with implied constant independent of $u \in C_c(\mathbb{R}^n)$.

By using the truncated cones Γ_x^a, we are only averaging over admissible balls in the definition of the operators. The idea is, of course, to exploit the doubling property of the gaussian measure on these balls. This makes the operators "admissible". The reader should notice, however, that they are not local, in the sense that their kernels are not supported in a region of the form $\{(x,y) \in \mathbb{R}^{2n} \; ; \; |x-y| \leq \frac{a}{1+|x|+|y|}\}$. Moreover, they can not be written as sums of local operators. This is due to a lack of off-diagonal estimates, that is a crucial difference between the Ornstein-Uhlenbeck semigroup and the heat semigroup.

2. A COVERING LEMMA

In this section we introduce partitions of $\mathbb{R}^n$ into "admissible" dyadic cubes and use them to prove a covering lemma which will be needed later on.

We begin with a brief discussion of admissible balls. Let

$$m(x) := \min\left\{1, \frac{1}{|x|}\right\}, \quad x \in \mathbb{R}^n.$$

For $a > 0$ we define

$$\mathscr{B}_a := \left\{B(x,r) \colon x \in \mathbb{R}^n, \ 0 \leq r \leq am(x)\right\}.$$

The balls in $\mathscr{B}_a$ are said to be *admissible at scale a*. It is a fundamental observation of Mauceri and Meda [9] that admissible balls enjoy a doubling property:

Lemma 2.1 (Doubling property). *Let $a, \tau > 0$. There exists a constant $d = d_{a,\tau,n}$, depending only on a, τ, and the dimension n, such that if $B_1 = B(c_1, r_1) \in \mathscr{B}_a$ and $B_2 = B(c_2, r_2)$ have non-empty intersection and $r_2 \leq \tau r_1$, then*

$$\gamma(B_2) \leq d\gamma(B_1).$$

In particular this lemma implies that for all $a > 0$ there exists a constant $d' = d'_a$ such that for all $B(x, r) \in \mathscr{B}_a$ we have

$$\gamma(B(x, 2r)) \leq d'\gamma(B(x, r)).$$

The first part of the next lemma, which is taken from [8], says, among other things, that if $B(x, r) \in \mathscr{B}_a$ and $|x - y| < br$, then $B(y, r) \in \mathscr{B}_c$ for some constant $c = c_{a,b}$ which depends only on a and b.

Lemma 2.2. *Let $a, b > 0$ be given.*

(i) *If $r \leq am(x)$ and $|x - y| \leq br$, then $r \leq c_{a,b}m(y)$, where $c_{a,b} := a(1 + ab)$.*
(ii) *If $|x - y| \leq bm(x)$, then $m(x) \leq (1 + b)m(y)$ and $m(y) \leq (2 + 2b)m(x)$.*

For $k \geq 0$ let Δ_k be the set of dyadic cubes at scale k, i.e.,

$$\Delta_k = \{2^{-k}(x + [0, 1)^n) \colon x \in \mathbb{Z}^n\}.$$

Following [8], in the gaussian case we only use cubes whose diameter depends on another parameter l, which keeps track of the distance from the ball to the origin. More precisely, define the *layers*

$$L_0 = [-1, 1)^n, \quad L_l = [-2^l, 2^l)^n \setminus [-2^{l-1}, 2^{l-1})^n \quad (l \geq 1),$$

and define, for $k, l \geq 0$,

$$\Delta_{k,l}^\gamma = \{Q \in \Delta_{l+k} \colon Q \subseteq L_l\}, \quad \Delta_k^\gamma = \bigcup_{l \geq 0} \Delta_{k,l}^\gamma, \quad \Delta^\gamma = \bigcup_{k \geq 0} \Delta_k^\gamma.$$

Note that if $Q \in \Delta_k^\gamma$ with $Q \subseteq L_l$, then its centre c_Q has norm $2^{l-1} \leq |c_Q| \leq 2^l\sqrt{n}$ and we have

$$(2.1) \qquad\qquad \mathrm{diam}(Q) = 2^{-k-l}\sqrt{n} \leq 2^{-k}n\,m(c_Q).$$

Lemma 2.3. *If a ball $B(x, r) \in \mathscr{B}_a$ intersects a cube $Q \in \Delta_0^\gamma$ with center c_Q, then*

$$r \leq 2a(a + n)m(c_Q).$$

Proof. We consider two cases. First, if $|c_Q| \geq 2(a + n)$, we notice that

$$r \leq \frac{a}{|x|} \leq \frac{a}{|c_Q| - (r + nm(c_Q)/2)} \leq \frac{a}{|c_Q| - (a + n/2)} \leq \frac{2a}{|c_Q|} = 2am(c_Q);$$

in the first inequality we used that $\mathrm{diam}(Q) \leq nm(c_Q)$ by (2.1), in the second we used that $m(c_Q) \leq 1$ and $r \leq am(x) \leq a$, the third follows from the assumption we made, and the final identity follows by noting that $|c_Q| \geq 2n \geq 1$. Second, if $|c_Q| \leq 2(a + n)$, then together with $1 \leq 2(a + n)$ we obtain $1 \leq 2(a + n)m(c_Q)$ and $r \leq a \leq 2a(a + n)m(c_Q)$. $\square$

We denote by $\alpha \circ Q$ the cube with the same centre as Q and α times its side-length; similar notation is used for balls. Cubes in Δ^γ enjoy the following doubling property:

Lemma 2.4. *Let $\alpha > 0$. There exists a constant $C_{\alpha,n}$, depending only on α and the dimension n, such that for every cube $Q \in \Delta^\gamma$ we have*

$$\gamma(\alpha \circ Q) \leq C_{\alpha,n}\gamma(Q).$$

Proof. Without loss of generality we may assume that $\alpha > 1$. Let $Q \in \Delta^\gamma_{k,l}$ with center y and side-length $2s$. Set $B = B(y, s)$ and note that $B \subseteq Q$. Moreover, we have $\alpha \circ Q \subseteq \alpha\sqrt{n} \circ B$. Since, if $|y| > 1$,

$$2s = \frac{\text{diam}(Q)}{\sqrt{n}} = 2^{-k-l} \leq 2^{-l} \leq \frac{\sqrt{n}}{|y|} = \sqrt{n}m(y),$$

and, if $|y| \leq 1$,

$$2s = 2^{-k-l} \leq 1 \leq \sqrt{n}m(y),$$

it follows that $B \in \mathscr{B}_{\sqrt{n}/2}$. Using the doubling property for admissible balls from Lemma 2.1 we now obtain

$$\gamma(\alpha \circ Q) \leq \gamma(\alpha\sqrt{n} \circ B) \leq C_{\alpha,n}\gamma(B) \leq C_{\alpha,n}\gamma(Q).$$

$\square$

Lemma 2.5. *Let $F \subseteq \mathbb{R}^n$ be a non-empty set, let $a, b, c > 0$ be fixed, and let*

$$O := \{x \in \mathbb{R}^n : 0 < d(x, F) \leq am(x)\}.$$

There exists a sequence $(x_k)_{k \geq 1}$ in O with the following properties:

(i) $O \subseteq \bigcup_{k \geq 1} B(x_k, bd(x_k, F))$;

(ii) $\sum_{k \geq 1} \gamma(B(x_k, cd(x_k, F))) \lesssim \gamma(O)$ *with constant depending only on a, b, c, n.*

Proof. Let $\delta := \min\{\frac{1}{2}, b\}$. We use a Whitney covering of

$$O' := \{z \in \mathbb{R}^n : 0 < d(z, F) < 2am(z)\}$$

by disjoint cubes Q_k such that

$$\tfrac{1}{4}\delta d(Q_k, \complement O') \leq \text{diam}(Q_k) \leq \delta d(Q_k, \complement O'),$$

(see [14, VI.1]). We discard the cubes that do not intersect O and relabel the remaining sequence of cubes as $(Q_k)_{k \geq 1}$ with centers $(c_k)_{k \geq 1}$. For each $k \geq 1$ pick $x_k \in O \cap Q_k$.

To check that the balls $B(c_k, \text{diam}(Q_k))$ are admissible, we use the fact that $\delta \leq \frac{1}{2}$ to obtain

$$|c_k - x_k| \leq \tfrac{1}{2}\text{diam}(Q_k) \leq \tfrac{1}{4}d(Q_k, \complement O') \leq \tfrac{1}{4}d(x_k, F) \leq \tfrac{1}{4}am(x_k).$$

Lemma 2.2(ii) then shows that $m(x_k) \leq (1 + \frac{a}{4})m(c_k)$. It follows that the balls $B(c_k, \text{diam}(Q_k))$ are admissible.

Next, $\text{diam}(Q_k) \leq \delta d(Q_k, \complement O') \leq bd(x_k, F)$, so (i) follows from

$$O \subseteq \bigcup_{k \geq 1} Q_k \subseteq \bigcup_{k \geq 1} B(x_k, \text{diam}(Q_k)) \subseteq \bigcup_{k \geq 1} B(x_k, bd(x_k, F)).$$

Towards the proof of (ii), we claim that for all $x \in O$,

$$d(x, F) \leq 3\max\{1, a\}d(x, \complement O').$$

To prove the claim, we fix $x \in O$ and pick an arbitrary $y \in \complement O'$. Setting $\varepsilon := \frac{1}{3} \min\{1, \frac{1}{a}\}$ we need to prove that

$$|x - y| \geq \varepsilon d(x, F).$$

From $y \notin O'$ we know that either $d(y, F) \geq 2am(y)$ or $d(y, F) = 0$. In the latter case we have $y \in \overline{F}$, hence $\varepsilon d(x, F) \leq d(x, F) \leq |x - y|$, so in what follows we may assume that $d(y, F) \geq 2am(y)$. From $x \in O$ we know that $d(x, F) \leq am(x)$. Suppose, for a contradiction, that $|x - y| < \varepsilon d(x, F)$. Then $|x - y| < \varepsilon am(x)$ and therefore $m(x) \leq (1 + \varepsilon a)m(y)$ by Lemma 2.2(ii). Also, for all $f \in F$ we have $|x - y| \geq |y - f| - |f - x| \geq 2am(y) - |f - x|$. Minimising over f, this gives $|x - y| \geq 2am(y) - d(x, F)$. Since also $\varepsilon d(x, F) > |x - y|$, we find that $am(y) < \frac{1}{2}(1 + \varepsilon)d(x, F) \leq \frac{1}{2}(1 + \varepsilon)am(x)$. It follows that $m(y) < \frac{1}{2}(1 + \varepsilon)m(x)$, and in combination with the inequality $m(x) \leq (1 + \varepsilon a)m(y)$ we get

$$2 < (1 + \varepsilon)(1 + \varepsilon a).$$

On the other hand, recalling that $\varepsilon = \frac{1}{3} \min\{1, \frac{1}{a}\}$ we see that $(1 + \varepsilon)(1 + \varepsilon a) \leq (1 + \frac{1}{3})(1 + \frac{1}{3}) = \frac{16}{9} < 2$. This contradicts the previous inequality and the claim is proved.

Combining the estimate

$$d(x_k, \complement O') \leq d(Q_k, \complement O') + \operatorname{diam}(Q_k) \leq \left(1 + \frac{4}{\delta}\right) \operatorname{diam}(Q_k)$$

with the claim, we obtain

$$d(x_k, F) \leq 3 \max\{1, a\} d(x_k, \complement O') \leq 3\left(1 + \frac{4}{\delta}\right) \max\{1, a\} \operatorname{diam}(Q_k).$$

Recalling the inequality $|c_k - x_k| \leq \frac{1}{4} d(x_k, F)$ proved before, and then using the doubling property in combination with the above inequality, we obtain

$$\sum_{k \geq 1} \gamma(B(x_k, cd(x_k, F))) \leq \sum_{k \geq 1} \gamma(B(c_k, (c + \tfrac{1}{4})d(x_k, F)))$$

$$\lesssim \sum_{k \geq 1} \gamma(B(c_k, \operatorname{diam}(Q_k))) \lesssim \sum_{k \geq 1} \gamma(Q_k) \leq \gamma(O').$$

To finish the proof we will show that $\gamma(O') \lesssim \gamma(O)$ with a constant depending only on a and n. Using the notation of Lemma 2.4 it suffices to show that there exists a sequence of disjoint cubes $(Q_i)_{i \geq 1} \subseteq \Delta^\gamma$ contained in O such that

$$O' \setminus O \subseteq \bigcup_{i \geq 1} M_{a,n} \circ Q_i$$

with $M_{a,n}$ depending only on a and n. Once this has been shown the claim follows from Lemma 2.4:

$$\gamma(O' \setminus O) \leq \sum_{i \geq 1} \gamma(M_{a,n} \circ Q_i) \lesssim \sum_{i \geq 1} \gamma(Q_i) = \gamma\left(\bigcup_{i \geq 1} Q_i\right) \leq \gamma(O)$$

and consequently $\gamma(O') \lesssim \gamma(O)$.

Let

$$O'' := \{x \in \mathbb{R}^n : \tfrac{1}{3}am(x) < d(x, F) < \tfrac{2}{3}am(x)\}.$$

Every point $x \in O''$ belongs to some maximal cube $Q_x \in \Delta^\gamma$ entirely contained in O. Since any two such maximal cubes are either equal or disjoint, we may select

a sequence $(x_i)_{i \geq 1}$ in O'' such that the maximal cubes $Q_{x_i} \in \Delta^\gamma$ are disjoint and cover O''. We will show that these cubes have the desired property.

Fix $y \in O' \setminus O$. Then $d(y, F) = cm(y)$ for some $a \leq c < 2a$. Choose $f \in \overline{F}$ with $|f - y| = cm(y)$ (this is possible since $\overline{F} \cap \{z \colon |y - z| \leq 2cm(y)\}$ is compact and non-empty). Choose $0 < \lambda < 1$ such that $g := (1 - \lambda)f + \lambda y$ belongs to O''. Choose the index i such that $g \in Q_{x_i}$.

Let $\beta \in (0, \frac{1}{3})$ be so small that $(\frac{2}{3} + \beta)(1 + \beta a) < 1$. We claim that $0 < d(z, F) < am(z)$ whenever $|z - g| < \beta am(g)$. To prove this, note that on the one hand,

$$d(z, F) \geq d(g, F) - |z - g| \geq \tfrac{1}{3}am(g) - \beta am(g) > 0,$$

while on the other hand, by Lemma 2.2,

$$d(z, F) \leq d(g, F) + |z - g| \leq \tfrac{2}{3}am(g) + \beta am(g) \leq (\tfrac{2}{3} + \beta)(1 + \beta a)am(z) < am(z),$$

which proves the claim. It thus follows that $B(g, \beta am(g))$ is contained in O. By maximality, this implies that the diameter of Q_{x_i} is at least $\frac{1}{2}\beta am(g)$. Hence the side-length l_i of Q_{x_i} is at least $\frac{1}{2\sqrt{n}}\beta am(g)$. On the other hand we have

$$|g - y| = (1 - \lambda)|f - y| < d(y, F) \leq 2am(y) \leq 2a(1 + 2a)m(g)$$

by Lemma 2.2. For any $M > 1$, the cube $M \circ Q_{x_i}$ contains the ball $B(g, \frac{1}{2}l_i(M - 1))$, and therefore $y \in M \circ Q_{x_i}$ provided $\frac{1}{2}l_i(M - 1) \geq 2a(1 + 2a)m(g)$. This happens if we take $M = 1 + 8\frac{\sqrt{n}}{\beta}(1 + 2a)$, since then

$$\tfrac{1}{2}l_i(M - 1) \geq \frac{1}{4\sqrt{n}}\beta am(g) \cdot 8\frac{\sqrt{n}}{\beta}(1 + 2a) = 2a(1 + 2a)m(g).$$

It thus follows that $M \circ Q_{x_i}$ contains y, which completes the proof. $\qquad\square$

3. Change of aperture for maximal functions

In the proof of Theorem 1.1 we need a change of aperture result for the admissible cone appearing in the definition of non-tangential maximal functions. For this purpose we define, for $A, a > 0$, the *non-tangential maximal function with parameters A, a* by

$$T^*_{(A,a)}u(x) := \sup_{(y,t) \in \Gamma_x^{(A,a)}(\gamma)} \left(\frac{1}{\gamma(B(y, At))} \int_{B(y, At)} |e^{-t^2 L}u(z)|^2 d\gamma(z) \right)^{\frac{1}{2}},$$

where

$$\Gamma_x^{(A,a)}(\gamma) := \left\{ (y, t) \in \mathbb{R}^n \times (0, \infty) \colon |y - x| < At \text{ and } t < am(x) \right\}$$

is the *admissible cone with parameters A, a* based at the point $x \in \mathbb{R}^n$. The parameter A is called the *aperture* of the cone.

In what follows we will fix the dimension n and write $L^p(\gamma) := L^p(\mathbb{R}^n; \gamma)$.

Theorem 3.1 (Change of aperture)**.** *For all $A, A', a > 0$ there exists a constant D, depending only on A, A', a, and the dimension n, such that for all $u \in L^1(\gamma)$ and $\sigma > 0$ we have*

$$\gamma\left(\left\{x \in \mathbb{R}^n \colon T^*_{(A,a)}u(x) > D\sigma\right\}\right) \lesssim \gamma\left(\left\{x \in \mathbb{R}^n \colon T^*_{(A',a')}u(x) > \sigma\right\}\right)$$

with $a' = a(1 + 2aA)(1 + A'a(1 + 2aA))$ and with implied constant independent of u and σ. In particular,

$$\|T^*_{(A,a)}u\|_{L^1(\gamma)} \lesssim \|T^*_{(A',a')}u\|_{L^1(\gamma)}$$

with implied constant independent of $u \in L^1(\gamma)$.

The proof of this theorem follows known arguments in the euclidean case [5]. We begin with a gaussian weak type $(1,1)$ estimate from [9]. For the convenience of the reader we include an alternative and self-contained proof.

Lemma 3.2. *Let $a > 0$. For $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ put*

$$M_a^* f(x) := \sup_{B(x,r) \in \mathscr{B}_a} \frac{1}{\gamma(B(x,r))} \int_{B(x,r)} |f(y)| \, d\gamma(y).$$

Then for all $\tau > 0$,

$$\tau \gamma(\{M_a^*(f) > \tau\}) \lesssim \|f\|_{L^1(\gamma)}$$

with implied constant only depending on a and n.

Proof. Fix $f \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ and decompose it as $f = \sum_{Q \in \Delta_0^\gamma} 1_Q f$. We denote by c_Q the centre of a cube Q. The idea of this proof is that the gaussian density is essentially equal to $e^{-\frac{1}{2}|c_Q|^2}$ on an admissible ball $B(c_Q, r_Q)$ and the support of $M_A^*(1_Q f)$ is included in such admissible balls.

To make this precise, consider a cube $Q \in \Delta_0^\gamma$, and suppose that a ball $B(x,r) \in \mathscr{B}_a$ intersects Q. Then Lemma 2.3 implies that $r \leq 2a(a+n)m(c_Q)$. As a consequence, for any $y \in B(x,r)$, we use the triangle inequality and (2.1) to obtain

$$|y - c_Q| \leq 2r + \tfrac{1}{2}\mathrm{diam}(Q) \leq (4a(a+n) + \tfrac{1}{2}n)m(c_Q) =: b_{a,n} m(c_Q),$$

and thus

$$\left||c_Q|^2 - |y|^2\right| \leq |c_Q + y|\,|c_Q - y| \leq \big(2|c_Q| + |c_Q - y|\big)|c_Q - y| \leq 2b_{a,n} + b_{a,n}^2.$$

This inequality implies that

$$e^{-\frac{1}{2}|y|^2} \eqsim e^{-\frac{1}{2}|c_Q|^2}$$

with implied constants depending only on a and n.

Using this estimate we obtain

$$(3.1) \qquad \frac{1}{\gamma(B(x,r))} \int_{B(x,r)} |1_Q(y)f(y)| d\gamma(y) \lesssim \frac{1}{|B(x,r)|} \int_{B(x,r)} |1_Q(y)f(y)| dy,$$

where $|B(x,r)|$ denotes the Lebesgue measure of the ball; the constants depend only on a and n.

Next we note that if $M_a^*(1_Q f)(x) > 0$, then there exists a ball $B \in \mathscr{B}_a$ such that $B \cap Q \neq \emptyset$. Using (3.1), we thus have, for all $\tau > 0$,

$$\gamma(\{M_a^*(1_Q f) > \tau\}) \eqsim \int_{\{M_a^*(1_Q f) > \tau\}} e^{-\frac{1}{2}|x|^2} dx \lesssim e^{-\frac{1}{2}|c_Q|^2} |\{M_a^*(1_Q f) > \tau\}|$$

$$\lesssim e^{-\frac{1}{2}|c_Q|^2} |\{M_{HL}(1_Q f) > \tau\}|,$$

where M_{HL} denotes the euclidean Hardy-Littlewood maximal operator. Using the weak type $(1,1)$ bound for the latter, we get

$$\tau \gamma(\{M_a^*(1_Q f) > \tau\}) \lesssim e^{-\frac{1}{2}|c_Q|^2} \int_Q |f(y)| dy \lesssim \|1_Q f\|_{L^1(\gamma)},$$

with constants depending only on a and n.

Now fix a cube $Q \in \Delta_0^\gamma$ with centre c_Q. For $x \in Q$ we have $|x - c_Q| \leq \frac{1}{2}\mathrm{diam}(Q) \leq \frac{1}{2}nm(c_Q)$ by (2.1), and therefore by the second part of Lemma 2.2(ii),

$m(x) \leq 2(1 + \frac{1}{2}n)m(c_Q)$. Hence if $B(x,r) \in \mathcal{B}_a$ and $x \in Q$, then $r \leq am(x) \leq (2+n)am(c_Q)$. Thus

$$B(x,r) \subseteq \bigcup_{l=1}^{N_Q} Q^{(l)},$$

where we denote by $Q^{(l)}$, $l = 1, ..., N_Q$, the cubes from Δ_0^γ that satisfy $d(Q, Q^{(l)}) \leq (2+n)am(c_Q)$. Remark that $N_Q \leq N$, where $N = N_{a,n}$ only depends on a and n.

It follows from the preceding considerations that

$$M_a^* f(x) \leq M_a^* \big(\sum_{l=1}^{N_Q} 1_{Q^{(l)}} f \big)(x) \leq \sum_{l=1}^{N_Q} M_a^*(1_{Q^{(l)}} f)(x)$$

for $x \in Q$, and thus

$$\gamma(\{M_a^* f > \tau\}) = \sum_{Q \in \Delta_0^\gamma} \gamma(\{M_a^* f > \tau\} \cap Q) \leq \sum_{Q \in \Delta_0^\gamma} \gamma\Big(\Big\{ \sum_{l=1}^{N_Q} M_a^*(1_{Q^{(l)}} f) > \tau \Big\}\Big)$$

$$\leq \sum_{Q \in \Delta_0^\gamma} \sum_{l=1}^{N_Q} \gamma\Big(\Big\{ M_a^*(1_{Q^{(l)}} f) > \frac{\tau}{N_Q} \Big\}\Big) \lesssim \sum_{Q \in \Delta_0^\gamma} \frac{N_Q}{\tau} \sum_{l=1}^{N_Q} \|1_{Q^{(l)}} f\|_{L^1(\gamma)}$$

$$\lesssim \frac{NN'}{\tau} \|f\|_{L^1(\gamma)},$$

with implied constant depending only on a and n; the N' in the last inequality accounts for the fact that, given $Q' \in \Delta_0^\gamma$, there are at most N' cubes $Q \in \Delta_0^\gamma$ such that $\text{dist}(Q, Q') \leq (2+n)am(c_Q)$, where again N' depends only on a and n. □

Proof of Theorem 3.1. It suffices to prove the inequality for test functions $u \in C_c(\mathbb{R}^n)$. For the rest of the proof we fix $u \in C_c(\mathbb{R}^n)$. Using the doubling property on admissible balls, we fix a constant $\tau > 0$ such that

$$\gamma(B(y, (A' + 4A)t)) < \frac{1}{\tau} \gamma(B(y, A't)) \quad \forall B(y,t) \in \mathscr{B}_{c_{a,2A}},$$

where $c_{a,2A} = (1 + 2aA)a$ is the constant arising from Lemma 2.2(i). For $\sigma > 0$ we define

$$E_\sigma := \{x \in \mathbb{R}^n : T^*_{(A',a')} u(x) > \sigma\},$$

$$\widetilde{E_\sigma} := \{x \in \mathbb{R}^n : M_b^*(1_{E_\sigma})(x) > \tau\},$$

where $M_b^* f$ is defined as in the lemma and $b := (A' + 2A)a$. The scheme of the proof is the following. We first prove (step 1) that, if $x \notin \widetilde{E_\sigma}$ and $(y,t) \in \Gamma_x^{(2A,a)}(\gamma)$, then $B(y, A't) \nsubseteq E_\sigma$. We then use this fact (step 2) to prove that

$$\frac{1}{\gamma(B(y, A't))} \int_{B(y,A't)} |e^{-t^2 L} u(\zeta)|^2 \, d\gamma(\zeta) \leq \sigma^2,$$

for all $(y,t) \in \Gamma_x^{(2A,a)}(\gamma)$ with $x \notin \widetilde{E_\sigma}$. This eventually gives (step 3) that there exists $D = D_{A,A',a,n} > 0$ such that $\{x \in \mathbb{R}^n ; T^*_{(A,a)} u(x) > D\sigma\} \subseteq \widetilde{E_\sigma}$. The proof is then concluded using Lemma 3.2 applied to 1_{E_σ}. In the estimates that follow, the implicit constants are independent of u and σ.

Throughout steps 1–3 below, we fix a point $x \notin \widetilde{E_\sigma}$ and a point $(y,t) \in \Gamma_x^{(2A,a)}(\gamma)$.

Step 1 – We claim that $B(y, A't) \not\subseteq E_\sigma$. To prove this, first note that from $|x - y| \leq 2At$ we have

$$B(y, A't) \subseteq B(x, (A' + 2A)t) \subseteq B(y, (A' + 4A)t).$$

Furthermore, $t \leq am(x)$, and therefore $B(x, (A' + 2A)t) \in \mathscr{B}_{(A'+2A)a} = \mathscr{B}_b$. If we now assume that the claim is false, we get

$$
\begin{aligned}
M_b^*(1_{E_\sigma})(x) &= \sup_{B(x,r) \in \mathscr{B}_b} \frac{\gamma(B(x,r) \cap E_\sigma)}{\gamma(B(x,r))} \\
&\geq \sup_{B(x,r) \in \mathscr{B}_b} \frac{\gamma(B(x,r) \cap B(y, A't))}{\gamma(B(x,r))} \\
&\geq \frac{\gamma(B(x, (A' + 2A)t) \cap B(y, A't))}{\gamma(B(x, (A' + 2A)t))} \\
&= \frac{\gamma(B(y, A't))}{\gamma(B(x, (A' + 2A)t))} \\
&\geq \frac{\gamma(B(y, A't))}{\gamma(B(y, (A' + 4A)t))} \\
&> \tau,
\end{aligned}
$$

where the second inequality uses that $B(x, (A'+2A)t) \in \mathscr{B}_b$ and the last one follows from the definition of the constant τ and the observation that $B(y,t) \in \mathscr{B}_{c_{a,2A}}$ by Lemma 2.2(i), using that $B(x,t) \in \mathscr{B}_a$ and $|x - y| \leq 2At$. This contradicts the fact that $x \notin \widetilde{E_\sigma}$ and the claim is proved.

Step 2 – Since $B(y, A't) \not\subseteq E_\sigma$, there exists $\widetilde{y} \in B(y, A't)$ such that $\widetilde{y} \notin E_\sigma$, that is,

$$(3.2) \qquad \sup_{(z,s) \in \Gamma_{\widetilde{y}}^{(A',a')}(\gamma)} \frac{1}{\gamma(B(z, A's))} \int_{B(z,A's)} |e^{-s^2 L} u(\zeta)|^2 \, d\gamma(\zeta) \leq \sigma^2.$$

Remark also that, since $B(y,t) \in \mathscr{B}_{c_{a,2A}}$ and thus $t \leq c_{a,2A} m(y)$, Lemma 2.2 implies that $t \leq a'm(\widetilde{y})$. Thus $(y,t) \in \Gamma_{\widetilde{y}}^{(A',a')}(\gamma)$ and therefore (3.2) implies

$$(3.3) \qquad \frac{1}{\gamma(B(y, A't))} \int_{B(y,A't)} |e^{-t^2 L} u(\zeta)|^2 \, d\gamma(\zeta) \leq \sigma^2.$$

Step 3 – Next let $(w,s) \in \Gamma_x^{(A,a)}(\gamma)$ be arbitrary and fixed for the moment. Then $w \in B(x, As)$. For any $v \in B(w, As)$ we have $|v - x| \leq |v - w| + |w - x| \leq 2As$. Since also $s \leq am(x)$, it follows that $(v,s) \in \Gamma_x^{(2A,a)}(\gamma)$. Also, since $|v - w| \leq As$ implies $B(v, A's) \subseteq B(w, (A' + A)s)$, we have

$$\gamma(B(v, A's)) \leq \gamma(B(w, (A' + A)s)) \lesssim \gamma(B(w, As))$$

by the doubling property for admissible balls; the balls $B(w, As)$ are indeed admissible by Lemma 2.2(i).

We can cover $B(w, As)$ with finitely many balls of the form $B(v_i, A's)$ with $v_i \in B(w, As)$; this can be achieved with $N = N(A, A', n)$ balls. We then have, by (3.3),

$$\frac{1}{\gamma(B(w, As))} \int_{B(w,As)} |e^{-s^2 L} u(z)|^2 \, d\gamma(z)$$

$$\lesssim \sum_{i=1}^{N} \frac{1}{\gamma(B(v_i, A's))} \int_{B(v_i, A's)} |e^{-s^2 L} u(z)|^2 d\gamma(z) \lesssim \sigma^2.$$

Taking the supremum over all $(w, s) \in \Gamma_x^{(A,a)}(\gamma)$, we infer that there exists a constant $D > 0$, depending only on A, A', a, and the dimension n, such that $T_{(A,a)}^* u(x) \leq D\sigma$ for all $x \notin \widetilde{E_\sigma}$.

We have now shown that $\{T_{(A,a)}^* u(x) > D\sigma\} \subseteq \widetilde{E_\sigma}$. The first assertion of the theorem follows from this via Lemma 3.2. The second assertion follows from the first by integration:

$$\|T_{(A,a)}^* u\|_{L^1(\gamma)} = D \int_0^\infty \gamma(\{x \in \mathbb{R}^n : T_{(A,a)}^* u(x) > D\sigma\}) \, d\sigma$$

$$\lesssim \int_0^\infty \gamma(\widetilde{E_\sigma}) \, d\sigma \lesssim \int_0^\infty \gamma(E_\sigma) \, d\sigma = \|T_{(A',a')}^* u\|_{L^1(\gamma)}.$$

Since the choice of $A, A', a \geq 0$ was arbitrary, this concludes the proof. $\qquad\square$

4. Proof of Theorem 1.1

In this section we follow the method pioneered in [5] for proving square function estimates in Hardy spaces. This method has recently been adapted in a variety of contexts (see [1, 2, 7]). Here, we modify the version given in [7] to avoid using the doubling property on non-admissible balls, and to take into account differences between the Laplace and the Ornstein-Uhlenbeck operators. As a typical example of the latter phenomenon, we start by proving a gaussian version of the parabolic Caccioppoli inequality. Recall that L is the Ornstein-Uhlenbeck operator, defined for $f \in C_b^2(\mathbb{R}^n)$ by

$$(4.1) \qquad\qquad Lf(x) = -\Delta f(x) + x \cdot \nabla f(x).$$

Note that, for all $f, g \in C_b^2(\mathbb{R}^n)$, one has the integration by parts formula

$$\int_{\mathbb{R}^n} Lf \cdot g \, d\gamma = \int_{\mathbb{R}^n} \nabla f \cdot \nabla g \, d\gamma$$

Lemma 4.1. *Let* $v : \mathbb{R}^n \times (0, \infty) \to \mathbb{C}$ *be a* $C^{1,2}$*-function such that* $v(\cdot, t) \in C_b^2(\mathbb{R}^n)$ *for all* $t > 0$*, and suppose that*

$$\partial_t v + Lv = 0$$

on $I(x_0, t_0, 2r) := B(x_0, 2cr) \times [t_0 - 4r^2, t_0 + 4r^2]$ *for some* $r \in (0, 1)$, $0 < C_0 \leq c \leq C_1 < \infty$*, and* $t_0 > 4r^2$*. Then*

$$\int_{I(x_0,t_0,r)} |\nabla v(x, t)|^2 \, d\gamma(x) \, dt \lesssim \frac{1 + r|x_0|}{r^2} \int_{I(x_0,t_0,2r)} |v(x, t)|^2 \, d\gamma(x) \, dt,$$

with implied constant depending only on the dimension n*,* C_0 *and* C_1*.*

Proof. Let $\eta \in C^\infty(\mathbb{R}^n \times (0, \infty))$ be a cut-off function such that $0 \leq \eta \leq 1$ on $\mathbb{R}^n \times (0, \infty)$, $\eta \equiv 1$ on $I(x_0, t_0, r)$, $\eta \equiv 0$ on the complement of $I(x_0, t_0, 2r)$, and

$$\|\nabla \eta\|_\infty \lesssim \frac{1}{r}, \quad \|\partial_t \eta\|_\infty \lesssim \frac{1}{r^2}, \quad \|\Delta \eta\|_\infty \lesssim \frac{1}{r^2}$$

with implied constants depending only on n, C_0, C_1. Then, in view of $\|x \cdot \nabla \eta\|_\infty \lesssim$ $(|x_0| + 2r) \cdot \frac{1}{r}$ and recalling that $0 < r < 1$,

$$(4.2) \qquad \|L\eta\|_\infty \lesssim \frac{1}{r^2} + \frac{1}{r}|x_0| + 1 \lesssim \frac{1 + r|x_0|}{r^2},$$

where the implied constants depend only on n, C_0, C_1.

Considering real and imaginary parts separately, we may assume that all functions are real-valued. Integrating the identity

$$(\eta \nabla v) \cdot (\eta \nabla v) = (v \nabla \eta - \nabla(v\eta)) \cdot (v \nabla \eta - \nabla(v\eta))$$

and then using that

$$\int_{I(x_0,t_0,2r)} \eta^2 \nabla(v\eta) \cdot \nabla(v\eta) \, d\gamma \, dt \le \int_0^\infty \int_{\mathbb{R}^d} \nabla(v\eta) \cdot \nabla(v\eta) \, d\gamma \, dt$$
$$= \int_0^\infty \int_{\mathbb{R}^n} v\eta L(v\eta) \, d\gamma \, dt$$
$$= \int_{I(x_0,t_0,2r)} v\eta L(v\eta) \, d\gamma \, dt,$$

we obtain

$$\int_{I(x_0,t_0,r)} |\nabla v|^2 \, d\gamma \, dt \le \int_{I(x_0,t_0,2r)} \eta^2 |\eta \nabla v|^2 \, d\gamma \, dt$$
$$\le \int_{I(x_0,t_0,2r)} \eta^2 |v \nabla \eta|^2 \, d\gamma \, dt$$
$$(4.3) \qquad\qquad + \left| \int_{I(x_0,t_0,2r)} 2v\eta^2 \nabla(v\eta) \cdot \nabla \eta \, d\gamma \, dt \right|$$
$$+ \left| \int_{I(x_0,t_0,2r)} v\eta L(v\eta) \, d\gamma \, dt \right|.$$

For the first term on the right-hand side we have the estimate

$$\int_{I(x_0,t_0,2r)} \eta^2 |v \nabla \eta|^2 \, d\gamma \, dt \lesssim \frac{1}{r^2} \int_{I(x_0,t_0,2r)} |v|^2 \, d\gamma \, dt.$$

For the second term we have, by (4.2),

$$\left| \int_{I(x_0,t_0,2r)} 2v\eta^2 \nabla(v\eta) \cdot \nabla \eta \, d\gamma \, dt \right| = \frac{1}{2} \left| \int_{I(x_0,t_0,2r)} \nabla(v\eta)^2 \cdot \nabla \eta^2 \, d\gamma \, dt \right|$$
$$\le \frac{1}{2} \left| \int_{\mathbb{R}^n} (v\eta)^2 L\eta^2 \, d\gamma \, dt \right|$$
$$\lesssim \frac{1 + r|x_0|}{r^2} \int_{I(x_0,t_0,2r)} |v|^2 \, d\gamma \, dt$$

where we used the fact that η^2 satisfies the same assumptions as η and (4.2) was applied to η^2. To estimate the third term on the right-hand side of (4.3) we substitute the identity

$$L(v\eta) = \eta L v + v L\eta - 2\nabla v \cdot \nabla \eta = -\eta \partial_t v + v L\eta - 2\nabla v \cdot \nabla \eta$$

and estimate each of the resulting integrals:

$$\left| \int_{I(x_0,t_0,2r)} v\eta^2 \partial_t v \, d\gamma \, dt \right| = \frac{1}{2} \left| \int_{I(x_0,t_0,2r)} \eta^2 \partial_t v^2 \, d\gamma \, dt \right|$$

$$= \frac{1}{2} \Big| \int_{I(x_0,t_0,2r)} v^2 \partial_t \eta^2 \, d\gamma \, dt \Big|$$

$$= \Big| \int_{I(x_0,t_0,2r)} v^2 \eta \partial_t \eta \, d\gamma \, dt \Big|$$

$$\lesssim \frac{1}{r^2} \int_{I(x_0,t_0,2r)} |v|^2 \, d\gamma \, dt,$$

$$\Big| \int_{I(x_0,t_0,2r)} v^2 \eta L \eta \, d\gamma \, dt \Big| \lesssim \frac{1+r|x_0|}{r^2} \int_{I(x_0,t_0,2r)} |v|^2 \, d\gamma \, dt,$$

$$\Big| \int_{I(x_0,t_0,2r)} v \eta \nabla v \cdot \nabla \eta \, d\gamma \, dt \Big| = \frac{1}{4} \Big| \int_{I(x_0,t_0,2r)} \nabla v^2 \cdot \nabla \eta^2 \, d\gamma \, dt \Big|$$

$$= \frac{1}{4} \Big| \int_{\mathbb{R}^n} v^2 L \eta^2 \, d\gamma \, dt \Big|$$

$$\lesssim \frac{1+r|x_0|}{r^2} \int_{I(x_0,t_0,2r)} |v|^2 \, d\gamma \, dt.$$

$\square$

Below we shall apply the lemma with $v(x,t) = e^{-tL}u(x)$, where u is a function in $C_c(\mathbb{R}^n)$. From the representation $e^{-tL}u(x) = \int_{\mathbb{R}^n} M_t(x,y)u(y)\,dy$ where M is the Mehler kernel (see, e.g., [13]), it follows that v satisfies the differentiability and boundedness assumptions of the lemma.

We can now prove the main result of this paper. Recall that

$$S_a u(x) = \Big(\int_{\Gamma_x^{(1,a)}(\gamma)} \frac{1}{\gamma(B(y,t))} |t\nabla e^{-t^2 L}u(y)|^2 \, d\gamma(y) \frac{dt}{t} \Big)^{\frac{1}{2}}$$

$$= \Big(\int_{\mathbb{R}^n \times (0,\infty)} \frac{1_{B(x,t)}(y)}{\gamma(B(y,t))} 1_{(0,am(x))}(t) |t\nabla e^{-t^2 L}u(y)|^2 \, d\gamma(y) \frac{dt}{t} \Big)^{\frac{1}{2}}.$$

It will be convenient to define, for $\varepsilon > 0$,

$$S_a^\varepsilon u(x) := \Big(\int_{\mathbb{R}^n \times (0,\infty)} \frac{1_{B(x,t)}(y)}{\gamma(B(y,t))} 1_{(\varepsilon,am(x))}(t) |t\nabla e^{-t^2 L}u(y)|^2 \, d\gamma(y) \frac{dt}{t} \Big)^{\frac{1}{2}}.$$

Proof of Theorem 1.1. As in the proof of Lemma 4.1 it suffices to consider real-valued $u \in C_c(\mathbb{R}^n)$. Throughout the proof we fix $a > 0$ and set $K := c_{a,1}$ and $\widetilde{K} := c_{1+2K,2}$ using the notations of Lemma 2.2.

Let $F \subseteq \mathbb{R}^n$ be an arbitrary closed set and define

$$F^* := \big\{ x \in \mathbb{R}^n \colon \gamma(F \cap B(x,r)) \geq \tfrac{1}{2}\gamma(B(x,r)) \ \forall r \in (0, \widetilde{K}m(x)] \big\}.$$

Note that, since F is closed, $F^* \subseteq F$. For $0 < \varepsilon < 1$ and $1 < \alpha < 2$ put

$$R_\alpha^\varepsilon(F^*) := \{(y,t) \in \mathbb{R}^n \times (0,\infty) \colon d(y,F^*) < \alpha t \text{ and } t \in (\alpha^{-1}\varepsilon, \alpha K m(y))\}$$

and let $\partial R_\alpha^\varepsilon(F^*)$ be its topological boundary. As in [5, page 162] and [14, page 206] we may regularise this set and thus assume it admits a surface measure $d\sigma_\alpha^\varepsilon(y,t)$. Applying first Green's formula in $\mathbb{R}^n$ to the section of $R_\alpha^\varepsilon(F^*)$ at level t and using the definition of L (see (4.1)), and subsequently the fundamental theorem of calculus in the t-variable, we obtain the estimate

$$\int_{F^*} |S_a^\varepsilon u(x)|^2 \, d\gamma(x)$$

$$= \int_{\mathbb{R}^n \times (0,\infty)} \int_{F^*} \frac{1_{B(x,t)}(y)}{\gamma(B(y,t))} 1_{(\varepsilon, am(x))}(t) |t\nabla e^{-t^2 L} u(y)|^2 \, d\gamma(x) \, d\gamma(y) \frac{dt}{t}$$

$$\overset{(i)}{\leq} \int_{\mathbb{R}^n \times (0,\infty)} \int_{F^*} \frac{1_{B(y,t)}(x)}{\gamma(B(y,t))} 1_{(\varepsilon, Km(y))}(t) |t\nabla e^{-t^2 L} u(y)|^2 \, d\gamma(x) \, d\gamma(y) \frac{dt}{t}$$

$$\overset{(ii)}{=} \int_{\mathbb{R}^n \times (0,\infty)} \frac{\gamma(B(y,t) \cap F^*))}{\gamma(B(y,t))} 1_{\{d(y,F^*)<t\}} 1_{(\varepsilon, Km(y))}(t) |t\nabla e^{-t^2 L} u(y)|^2 \, d\gamma(y) \frac{dt}{t}$$

$$\leq \int_{\mathbb{R}^n \times (0,\infty)} 1_{\{d(y,F^*)<t\}} 1_{(\varepsilon, Km(y))}(t) |t\nabla e^{-t^2 L} u(y)|^2 \, d\gamma(y) \frac{dt}{t}$$

$$\leq \int_{R_\alpha^\varepsilon(F^*)} |t\nabla e^{-t^2 L} u(y)|^2 \, d\gamma(y) \frac{dt}{t}$$

$$\lesssim \int_{R_\alpha^\varepsilon(F^*)} tL e^{-t^2 L} u(y) \cdot e^{-t^2 L} u(y) \, d\gamma(y) \, dt$$

$$\qquad + \int_{\partial R_\alpha^\varepsilon(F^*)} |t\nabla e^{-t^2 L} u \cdot \nu^\parallel(y,t)| |e^{-t^2 L} u(y)| e^{-\frac{1}{2}|y|^2} \, d\sigma_\alpha^\varepsilon(y,t)$$

$$\lesssim \int_{R_\alpha^\varepsilon(F^*)} -\partial_t |e^{-t^2 L} u(y)|^2 \, d\gamma(y) \, dt$$

$$\qquad + \int_{\partial R_\alpha^\varepsilon(F^*)} |t\nabla e^{-t^2 L} u(y)| |e^{-t^2 L} u(y)| e^{-\frac{1}{2}|y|^2} \, d\sigma_\alpha^\varepsilon(y,t)$$

$$\lesssim \int_{\partial R_\alpha^\varepsilon(F^*)} |e^{-t^2 L} u(y) \nu^\perp(y,t)|^2 e^{-\frac{1}{2}|y|^2} \, d\sigma_\alpha^\varepsilon(y,t)$$

$$\qquad + \int_{\partial R_\alpha^\varepsilon(F^*)} |t\nabla e^{-t^2 L} u(y)| |e^{-t^2 L} u(y)| e^{-\frac{1}{2}|y|^2} \, d\sigma_\alpha^\varepsilon(y,t).$$

In the above computation, $\nu^\parallel$ denotes the projection of the normal vector ν to R_α^ε onto $\mathbb{R}^n$ and $\nu^\perp$ the projection of ν in the t direction. In step (i) we used that $1_{B(x,t)}(y) = 1_{B(y,t)}(x)$ and that $|x - y| < t$ and $t < am(x)$ imply $t < Km(y)$ via Lemma 2.2(i); in step (ii) we used that $B(y,t) \cap F^* \neq \emptyset$ implies $d(y, F^*) < t$. Of course, all implied constants in the above inequalities are independent of F, ε, α, and u.

If $(y,t) \in \partial R_\alpha^\varepsilon(F^*)$, then either $d(y, F^*) = \alpha t$ and $t \in [\alpha^{-1}\varepsilon, \alpha Km(y)]$, or else $d(y, F^*) < \alpha t$ and $t \in \{\alpha^{-1}\varepsilon, \alpha Km(y)\}$. By examining these three cases separately, each time distinguishing between the possible relative positions of $m(y)$ with respect to the numbers $\frac{1}{2}\varepsilon$, $\alpha^{-1}\varepsilon$, and ε, one checks that $\partial R_\alpha^\varepsilon(F^*) \subseteq \widetilde{B}^\varepsilon := \widetilde{B}_1^\varepsilon \cup \widetilde{B}_2^\varepsilon \cup \widetilde{B}_3^\varepsilon$ with

$$\widetilde{B}_1^\varepsilon := \{(y,t) \in \mathbb{R}^n \times (0,\infty) \colon t \in [\tfrac{1}{2}\varepsilon, \min\{\varepsilon, m(y)\}] \text{ and } d(y, F^*) \leq 2t\},$$

$$\widetilde{B}_2^\varepsilon := \{(y,t) \in \mathbb{R}^n \times (0,\infty) \colon t \in [\varepsilon, m(y)] \text{ and } t \leq d(y, F^*) \leq 2t\},$$

$$\widetilde{B}_3^\varepsilon := \{(y,t) \in \mathbb{R}^n \times (0,\infty) \colon t \in [m(y), 2Km(y)] \text{ and } d(y, F^*) \leq 2t\}.$$

Now notice that, on $\partial R_\alpha^\varepsilon(F^*)$, we have either $t = \frac{\varepsilon}{\alpha}$, $t = \alpha Km(y)$, or $t = \alpha^{-1} d(y, F^*)$. Integrating over $\alpha \in (1,2)$ with respect to $\frac{d\alpha}{\alpha}$ and changing variables using that $\frac{d\alpha}{\alpha} \sim \frac{dt}{t}$, we obtain

$$\int_{F^*} |S_a^\varepsilon u|^2 \, d\gamma \lesssim \int_{\widetilde{B}^\varepsilon} |e^{-t^2 L} u(y)|^2 d\gamma(y) \frac{dt}{t}$$

$$+ \left(\int_{\widetilde{B}^\varepsilon} |e^{-t^2 L} u(y)|^2 d\gamma(y) \frac{dt}{t} \right)^{\frac{1}{2}} \left(\int_{\widetilde{B}^\varepsilon} |t\nabla e^{-t^2 L} u(y)|^2 d\gamma(y) \frac{dt}{t} \right)^{\frac{1}{2}}$$

$$\lesssim \int_{\widetilde{B}^\varepsilon} |e^{-t^2 L} u(y)|^2 d\gamma(y) \frac{dt}{t} + \int_{\widetilde{B}^\varepsilon} |t\nabla e^{-t^2 L} u(y)|^2 d\gamma(y) \frac{dt}{t}.$$

Here, and in the estimates to follow, the implied constants are independent of F, ε, and u.

We have to estimate the following six integrals:

$$I_1 := \int_{\widetilde{B}_1^\varepsilon} |e^{-t^2 L} u(y)|^2 d\gamma(y) \frac{dt}{t}, \qquad I_2 := \int_{\widetilde{B}_1^\varepsilon} |t\nabla e^{-t^2 L} u(y)|^2 d\gamma(y) \frac{dt}{t},$$

$$I_3 := \int_{\widetilde{B}_2^\varepsilon} |e^{-t^2 L} u(y)|^2 d\gamma(y) \frac{dt}{t}, \qquad I_4 := \int_{\widetilde{B}_2^\varepsilon} |t\nabla e^{-t^2 L} u(y)|^2 d\gamma(y) \frac{dt}{t},$$

$$I_5 := \int_{\widetilde{B}_3^\varepsilon} |e^{-t^2 L} u(y)|^2 d\gamma(y) \frac{dt}{t}, \qquad I_6 := \int_{\widetilde{B}_3^\varepsilon} |t\nabla e^{-t^2 L} u(y)|^2 d\gamma(y) \frac{dt}{t}.$$

We start with I_1 and remark that, for $(y,t) \in \widetilde{B}_1^\varepsilon$, there exists $x \in F^*$ such that $|x-y| \leq 2t$. Since $t \leq \min\{\varepsilon, m(y)\} \leq m(y)$, by Lemma 2.2(i) we have $t \leq c_{1,2} m(x)$ and hence $t \leq \widetilde{K} m(x)$, noting that $c_{1,2} \leq c_{1+2K,2} = \widetilde{K}$. Therefore, by the definition of F^*,

$$(4.4) \qquad\qquad \gamma(F \cap B(x,t)) \geq \tfrac{1}{2}\gamma(B(x,t)).$$

(At this point the reader may wonder why F^* is defined in terms of $\widetilde{K}$ and not in terms of $c_{1,2}$. The reason is that the argument will be repeated in the estimation of I_2, I_5, and I_6; in the latter two cases, the definition of B_3^ε implies that one only gets $t \leq 2Km(y)$ and hence $t \leq c_{2K,2} m(x) \leq c_{1+2K,2} m(x)$). By (4.4) and doubling property for the admissible ball $B(x,t) \in \mathscr{B}_{c_{1,2}}$,

$$\gamma(F \cap B(y,3t)) \geq \gamma(F \cap B(x,t)) \geq \tfrac{1}{2}\gamma(B(x,t)) \gtrsim \gamma(B(x,3t)) \geq \gamma(B(y,t)),$$

and therefore

$$I_1 \lesssim \int_{\widetilde{B}_1^\varepsilon} \int_{F \cap B(y,3t)} \frac{1}{\gamma(B(y,t))} |e^{-t^2 L} u(y)|^2 \, d\gamma(z) \, d\gamma(y) \frac{dt}{t}$$

$$(4.5) \qquad \leq \int_{\mathbb{R}^n} \int_{\frac{1}{2}\varepsilon}^{\frac{1}{2}\varepsilon \vee \min\{\varepsilon, m(y)\}} \int_F \frac{1_{B(y,3t)}(z)}{\gamma(B(y,t))} |e^{-t^2 L} u(y)|^2 d\gamma(z) \frac{dt}{t} d\gamma(y)$$

$$\leq \int_F \int_{\frac{1}{2}\varepsilon}^{\frac{1}{2}\varepsilon \vee \min\{\varepsilon, c_{1,3} m(z)\}} \int_{B(z,3t)} \frac{1}{\gamma(B(y,t))} |e^{-t^2 L} u(y)|^2 \, d\gamma(y) \frac{dt}{t} d\gamma(z),$$

where in the last inequality we used that $t \leq m(y)$ and $|y-z| < 3t$ imply $t \leq c_{1,3} m(z)$ by Lemma 2.2(i).

Fix $(z,t) \in F \times (\frac{1}{2}\varepsilon, \frac{1}{2}\varepsilon \vee \min\{\varepsilon, c_{1,3} m(z)\})$. For all $y \in B(z,3t)$ we have $B(z,3t) \subseteq B(y,6t)$ and therefore, by the doubling property for $B(y,t)$ (noting that from $t < c_{1,3} m(z)$ and $|z-y| < 3t$ it follows that $t < c_{c_{1,3},3} m(y)$, so $B(y,t)$ is an admissible ball in $\mathscr{B}_{c_{c_{1,3},3}}$),

$$\int_{B(z,3t)} \frac{1}{\gamma(B(y,t))} |e^{-t^2 L} u(y)|^2 \, d\gamma(y) \lesssim \frac{1}{\gamma(B(z,3t))} \int_{B(z,3t)} |e^{-t^2 L} u(y)|^2 \, d\gamma(y)$$

$$\leq |T_{(3,c_{1,3})}^* u(z)|^2,$$

where the last inequality follows from $(z,t) \in \Gamma_z^{(3,c_{1,3})}(\gamma)$. Combining this with the previous inequality it follows that

$$I_1 \lesssim \int_F \int_{\frac{1}{2}\varepsilon}^{\varepsilon} |T^*_{(3,c_{1,3})}u(z)|^2 \frac{dt}{t} \, d\gamma(z) \lesssim \int_F |T^*_{(3,c_{1,3})}u(z)|^2 \, d\gamma(z).$$

We proceed similarly for I_2, using Lemma 4.1 to handle the gradient. With $\tau(z) := c_{1,3}m(z)$ we have, proceeding as in (4.5),

$$I_2 \lesssim \int_F \int_{\frac{1}{2}\varepsilon}^{\frac{1}{2}\varepsilon \vee \min\{\varepsilon,\tau(z)\}} \int_{B(z,3t)} \frac{1}{\gamma(B(y,t))} |t\nabla e^{-t^2 L}u(y)|^2 \, d\gamma(y) \frac{dt}{t} \, d\gamma(z)$$

$$\stackrel{(i)}{\lesssim} \int_{F \cap \{\tau(z) \geq \frac{1}{2}\varepsilon\}} \int_{\frac{1}{2}\varepsilon}^{\varepsilon} \frac{1}{\gamma(B(z,3\varepsilon))} \int_{B(z,3\varepsilon)} |t\nabla e^{-t^2 L}u(y)|^2 \, d\gamma(y) \frac{dt}{t} \, d\gamma(z)$$

$$\stackrel{(ii)}{\lesssim} \int_{F \cap \{\tau(z) \geq \frac{1}{2}\varepsilon\}} \sum_{l=2}^{7} \int_{\frac{l\varepsilon^2}{8}}^{\frac{(l+1)\varepsilon^2}{8}} \frac{1}{\gamma(B(z,3\varepsilon))} \int_{B(z,3\varepsilon)} |\nabla e^{-sL}u(y)|^2 \, d\gamma(y) \, ds \, d\gamma(z).$$

In (i) we used the inclusions $B(z,3t) \subseteq B(z,3\varepsilon) \subseteq B(z,6t) \subseteq B(y,9t)$ together with the doubling property for $B(y,t)$, and in (ii) we substituted $t^2 = s$.

For each $l \in \{2,\ldots,7\}$ we apply Lemma 4.1 with $t_0^l = \frac{1}{2}(\frac{l\varepsilon^2}{8} + \frac{(l+1)\varepsilon^2}{8}) = \frac{(2l+1)\varepsilon^2}{16}$, $c^l = 12$ and $(r^l)^2 = \frac{\varepsilon^2}{16}$. Together with the doubling property for $B(z,\varepsilon)$ (noting that $B(z,\varepsilon) \in \mathscr{B}_{2c_{1,3}}$ in view of $\varepsilon \leq 2t \leq 2c_{1,3}m(z)$), this gives

$$I_2 \lesssim \int_{F \cap \{\tau(z) \geq \frac{1}{2}\varepsilon\}} \sum_{l=2}^{7} \int_{\frac{(2l-3)\varepsilon^2}{16}}^{\frac{(2l+5)\varepsilon^2}{16}} \frac{1 + r^l|z|}{(r^l)^2}$$

$$\times \frac{1}{\gamma(B(z,6\varepsilon))} \int_{B(z,6\varepsilon)} |e^{-sL}u(y)|^2 \, d\gamma(y) \, ds \, d\gamma(z).$$

Fix $(z,s) \in (F \cap \{\tau(z) \geq \frac{1}{2}\varepsilon\}) \times (\frac{1}{16}\varepsilon^2, \frac{19}{16}\varepsilon^2)$. Then from $B(z,6\varepsilon) \subseteq B(z,24\sqrt{s}) \subseteq B(z,30\varepsilon)$ and the doubling property for the balls $B(z,\varepsilon) \in \mathscr{B}_{2c_{1,3}}$ (note that $\varepsilon \leq 2\tau(z) = 2c_{1,3}m(z)$),

$$\frac{1}{\gamma(B(z,6\varepsilon))} \int_{B(z,6\varepsilon)} |e^{-sL}u(y)|^2 \, d\gamma(y)$$

$$\lesssim \frac{1}{\gamma(B(z,24\sqrt{s}))} \int_{B(z,24\sqrt{s})} |e^{-sL}u(y)|^2 \, d\gamma(y) \leq |T^*_{(24,4c_{1,3})}u(z)|^2,$$

where the last step follows from $(z,\sqrt{s}) \in \Gamma_z^{(24,4c_{1,3})}(\gamma)$. Combining this with the previous estimate we obtain

$$I_2 \lesssim \int_F \sum_{l=2}^{7} \int_{\frac{(2l-3)\varepsilon^2}{16}}^{\frac{(2l+5)\varepsilon^2}{16}} \frac{1 + r^l|z|}{(r^l)^2} |T^*_{(24,4c_{1,3})}u(z)|^2 \, ds \, d\gamma(z)$$

$$\lesssim \int_F (1 + \varepsilon|z|)|T^*_{(24,4c_{1,3})}u(z)|^2 \, d\gamma(z),$$

where the last step follows from the fact that $r^l = \frac{1}{4}\varepsilon$.

We proceed with an estimate for I_3. Let

$$G := \{y \in \mathbb{R}^n : 0 < d(y,F^*) \leq 2m(y)\}.$$

Using Lemma 2.5, we cover G with a sequence of balls $B(x_k, r_k)$ with $x_k \in G$ and $r_k = \frac{1}{4}d(x_k, F^*)$ for all k, and

$$(4.6) \qquad \sum_{k \geq 1} \gamma(B(x_k, d(x_k, F^*))) \lesssim \gamma(G) \leq \gamma(\complement F^*).$$

with implied constant independent of u and F. Note that $B(x_k, r_k) \in \mathscr{B}_{\frac{1}{2}}$ for all k.

If $(y, t) \in \widetilde{B}_2^\varepsilon$, then $y \in G$ and therefore $y \in B(x_k, r_k)$ for some k, and $\frac{1}{2}d(y, F^*) \leq t \leq d(y, F^*)$. It follows that

$$
\begin{aligned}
I_3 &\leq \sum_k \int_{B(x_k, r_k)} \int_{\frac{1}{2}d(y, F^*)}^{d(y, F^*)} |e^{-t^2 L} u(y)|^2 \frac{dt}{t} \, d\gamma(y) \\
(4.7) \qquad &\leq \sum_k \int_{B(x_k, r_k)} \int_{\frac{1}{4}d(x_k, F^*)}^{\frac{5}{4}d(x_k, F^*)} |e^{-t^2 L} u(y)|^2 \frac{dt}{t} \, d\gamma(y) \\
&\leq \sum_k \int_{\frac{1}{4}d(x_k, F^*)}^{\frac{5}{4}d(x_k, F^*)} \int_{B(x_k, t)} |e^{-t^2 L} u(y)|^2 \, d\gamma(y) \frac{dt}{t}.
\end{aligned}
$$

In the second inequality we used that $y \in B(x_k, r_k)$ implies $|x_k - y| < r_k = \frac{1}{4}d(x_k, F^*)$, and the third inequality follows from Fubini's theorem and the inequality $r_k = \frac{1}{4}d(x_k, F^*) \leq \frac{1}{2}d(y, F^*) \leq t$.

Fix an index k and a number $t \in (\frac{1}{4}d(x_k, F^*), \frac{5}{4}d(x_k, F^*))$. Since F^* is contained in the closure of F we may pick $z_k \in F$ such that $|x_k - z_k| < 2d(x_k, F^*)$. By the choice of t this implies $|x_k - z_k| < 8t$. Since by assumption we have $t \leq \frac{5}{4}d(x_k, F^*) \leq \frac{5}{2}m(x_k)$ (the second inequality being a consequence of $x_k \in G$), and since $|x_k - z_k| < 8t$, from Lemma 2.2 we conclude that $t \leq dm(z_k)$ with $d := c_{\frac{5}{2}, 8}$. We conclude that $(x_k, t) \in \Gamma_{z_k}^{(8, d)}(\gamma)$ (since by definition this means that $|x_k - z_k| \leq 8t \leq 8dm(z_k)$) and consequently, using the doubling property for the admissible ball $B(x_k, t) \in \mathscr{B}_{\frac{5}{2}}$,

$$
\begin{aligned}
&\frac{1}{\gamma(B(x_k, t))} \int_{B(x_k, t)} |e^{-t^2 L} u(y)|^2 \, d\gamma(y) \\
&\qquad \lesssim \frac{1}{\gamma(B(x_k, 8t))} \int_{B(x_k, 8t)} |e^{-t^2 L} u(y)|^2 \, d\gamma(y) \leq |T_{(8, d)}^* u(z_k)|^2.
\end{aligned}
$$

Combining this with the previous inequalities we obtain

$$
\begin{aligned}
I_3 &\lesssim \left(\sup_{z \in F} |T_{(8, d)}^* u(z)|^2 \right) \sum_k \int_{\frac{1}{4}d(x_k, F^*)}^{\frac{5}{4}d(x_k, F^*)} \gamma(B(x_k, t)) \frac{dt}{t} \\
&\lesssim \left(\sup_{z \in F} |T_{(8, d)}^* u(z)|^2 \right) \sum_k \gamma(B(x_k, \tfrac{5}{4}d(x_k, F^*))) \\
&\lesssim \left(\sup_{z \in F} |T_{(8, d)}^* u(z)|^2 \right) \gamma(\complement F^*),
\end{aligned}
$$

where the last step used (4.6) and the doubling property (recall that $d(x_k, F^*) \leq 2m(x_k)$, so the balls $B(x_k, d(x_k, F^*))$ belong to $\mathscr{B}_2$).

For estimating I_4, we let G and $B(x_k, r_k)$ be as in the previous estimate. Proceeding as in the first two lines of (4.7) and applying the Fubini theorem, we get

$$
I_4 \lesssim \sum_k \int_{\frac{1}{4}d(x_k, F^*)}^{\frac{5}{4}d(x_k, F^*)} \int_{B(x_k, r_k)} |t\nabla e^{-t^2 L} u(y)|^2 \, d\gamma(y) \, \frac{dt}{t}
$$

$$
= \frac{1}{2} \sum_k \sum_{l=2}^{49} \int_{\frac{2l}{64}d^2(x_k, F^*)}^{\frac{2l+2}{64}d^2(x_k, F^*)} \int_{B(x_k, r_k)} |\nabla e^{-sL} u(y)|^2 \, d\gamma(y) \, ds.
$$

By Lemma 4.1, applied with $t_0 = \frac{2l+1}{64}d^2(x_k, F^*)$, $c = 2$ and $r = \frac{1}{8}d(x_k, F^*)$, this gives the estimate

$$
I_4 \lesssim \sum_k \sum_{l=2}^{49} \int_{\frac{2l-3}{64}d^2(x_k, F^*)}^{\frac{2l+5}{64}d^2(x_k, F^*)} \frac{1 + d(x_k, F^*)|x_k|}{d^2(x_k, F^*)} \int_{B(x_k, \frac{1}{2}d(x_k, F^*))} |e^{-sL} u(y)|^2 \, d\gamma(y) \, ds
$$

$$
\leq \sum_k \sum_{l=2}^{49} \int_{\frac{2l-3}{64}d^2(x_k, F^*)}^{\frac{2l+5}{64}d^2(x_k, F^*)} \frac{3}{d^2(x_k, F^*)} \int_{B(x_k, 4\sqrt{s})} |e^{-sL} u(y)|^2 \, d\gamma(y) \, ds,
$$

where we used that $d(x_k, F^*) \leq 2m(x_k) \leq \frac{2}{|x_k|}$ and that $s \geq \frac{1}{64}d^2(x_k, F^*)$ implies $\frac{1}{2}d(x_k, F^*) \leq 4\sqrt{s}$.

Fix k and pick an element $z_k \in F$ such that $|x_k - z_k| < 2d(x_k, F^*)$. Then for all s in the range of integration we have $|x_k - z_k| < 16\sqrt{s}$. Since $\sqrt{s} \leq \frac{3}{2}d(x_k, F^*) \leq 3m(x_k)$, from Lemma 2.2 we conclude that $\sqrt{s} \leq dm(z_k)$ with $d := c_{3,16}$. We conclude that $(x_k, 4\sqrt{s}) \in \Gamma_{z_k}^{(4, 4d)}(\gamma)$. This gives

$$
I_4 \lesssim \left(\sup_{z \in F} |T_{(4, 4d)}^* u(z)|^2 \right) \sum_k \frac{1}{d^2(x_k, F^*)} \int_{\frac{1}{64}d^2(x_k, F^*)}^{\frac{103}{64}d^2(x_k, F^*)} \gamma(B(x_k, 4\sqrt{s})) \, ds
$$

$$
\lesssim \left(\sup_{z \in F} |T_{(4, 4d)}^* u(z)|^2 \right) \sum_k \gamma(B(x_k, \tfrac{1}{2}\sqrt{103}d(x_k, F^*)))
$$

$$
\lesssim \left(\sup_{z \in F} |T_{(4, 4d)}^* u(z)|^2 \right) \sum_k \gamma(B(x_k, d(x_k, F^*)))
$$

$$
\lesssim \left(\sup_{z \in F} |T_{(4, 4d)}^* u(y)|^2 \right) \gamma(\complement F^*),
$$

where the second last step used the doubling property for admissible balls (recalling that $B(x_k, d(x_k, F^*)) \in \mathscr{B}_2$), and the last step used (4.6).

To estimate I_5, we proceed as we did for I_1:

$$
I_5 \lesssim \int_{\widetilde{B}_3^\varepsilon} \int_{F \cap B(y, 3t)} \frac{1}{\gamma(B(y, t))} |e^{-t^2 L} u(y)|^2 \, d\gamma(z) \, d\gamma(y) \, \frac{dt}{t}
$$

$$
\leq \int_{\mathbb{R}^n} \int_{m(y)}^{2Km(y)} \int_F \frac{1_{B(y, 3t)}(z)}{\gamma(B(y, t))} |e^{-t^2 L} u(y)|^2 \, d\gamma(z) \, \frac{dt}{t} \, d\gamma(y)
$$

$$
\overset{(i)}{\leq} \int_F \int_{(1+3c_{2K,3})^{-1}m(z)}^{c_{2K,3}m(z)} \int_{B(z, 3t)} \frac{1}{\gamma(B(y, t))} |e^{-t^2 L} u(y)|^2 \, d\gamma(y) \, \frac{dt}{t} \, d\gamma(z)
$$

$$
\lesssim \int_F |T_{(2K, c_{2K,3})}^* u(z)|^2 \, d\gamma(z),
$$

where in step (i) we used that $m(y) \le t \le 2Km(y)$ and $|y - z| < 3t$ imply $t \le c_{2K,3}m(z)$ by Lemma 2.2(i), so $|y - z| < 3c_{2K,3}m(z)$, and by an application of Lemma 2.2(ii) the latter implies $m(z) \le (1 + 3c_{2K,3})m(y) \le (1 + 3c_{2K,3})t$.

Finally we turn to I_6, which is treated as I_2. With $c = c_{2K,3}$ and $d = (1 + 3c_{2K,3})^{-1}$ as in the previous estimate, and using Lemma 4.1 as in the estimate for I_2, we get

$$
\begin{aligned}
I_6 &\lesssim \int_F \int_{dm(z)}^{cm(z)} \frac{1}{\gamma(B(z,3t))} \int_{B(z,3t)} |t\nabla e^{-t^2 L} u(y)|^2 \, d\gamma(y) \frac{dt}{t} \, d\gamma(z) \\
&= \frac{1}{2} \int_F \int_{d^2 m(z)^2}^{c^2 m(z)^2} \frac{1}{\gamma(B(z,3t))} \int_{B(z,3t)} |\nabla e^{-sL} u(y)|^2 \, d\gamma(y) \, ds \, d\gamma(z) \\
&\lesssim \int_F (1 + m(z)|z|) |T^*_{(A,a)} u(z)|^2 \, d\gamma(z) \\
&\lesssim \int_F |T^*_{(A,a)} u(z)|^2 \, d\gamma(z),
\end{aligned}
$$

for certain A, a independent of u, F, and ε.

Combining all these estimates, we obtain six couples $(A^{(j)}, a^{(j)})$ $(j = 1, ..., 6)$, and, passing to the limit $\varepsilon \downarrow 0$, the following estimate, valid for arbitrary closed subsets $F \subseteq \mathbb{R}^n$:

$$
\begin{aligned}
(4.8) \quad & \int_{F^*} |S_a u(x)|^2 \, d\gamma(x) \\
& \lesssim \sum_{j=1}^6 \left(\left(\sup_{z \in F} |T^*_{(A^{(j)},a^{(j)})} u(z)|^2 \right) \gamma(\complement F^*) + \int_F |T^*_{(A^{(j)},a^{(j)})} u(z)|^2 \, d\gamma(z) \right),
\end{aligned}
$$

with constants independent of F and u.

To finish the proof, we consider the distribution functions

$$
\gamma_{S_a u}(\sigma) := \gamma\big(\{x \in \mathbb{R}^n \colon S_a u(x) > \sigma\}\big),
$$
$$
\gamma_{T^*_{(A^{(j)},a^{(j)})} u}(\sigma) := \gamma\big(\{x \in \mathbb{R}^n \colon T^*_{(A^{(j)},a^{(j)})} u(x) > \sigma\}\big), \quad j = 1, \dots, 6.
$$

We fix $\sigma > 0$ for the moment, and apply (4.8) to the set

$$
F_\sigma := \big\{z \in \mathbb{R}^n \colon T^*_{(A^{(j)},a^{(j)})} u(z) \le \sigma, \; j = 1, \dots, 6\big\},
$$

and claim that $\complement F_\sigma^* \subseteq \{M^*_{\widetilde{K}}(1_{\complement F_\sigma}) > \frac{1}{2}\}$. Indeed, let $x \in \complement F_\sigma^*$ and fix $r \in (0, \widetilde{K}m(x)]$ such that $\gamma(B(x,r) \cap F_\sigma) < \frac{1}{2}\gamma(B(x,r))$. Then

$$
M^*_{\widetilde{K}}(1_{\complement F_\sigma})(x) \ge \frac{\gamma(B(x,r) \cap \complement F_\sigma)}{\gamma(B(x,r))} > \frac{1}{2},
$$

proving the claim.

Lemma 3.2 (with admissibility parameter $\widetilde{K}$, $\tau = \frac{1}{2}$, applied to the function $1_{\complement F_\sigma}$) gives us $\gamma(\complement F_\sigma^*) \lesssim \gamma(\complement F_\sigma)$. Using this in combination with the definition of F_σ, for $j = 1, \dots, 6$ we obtain

$$
\frac{1}{\sigma^2} \Big(\sup_{z \in F_\sigma} |T^*_{A^{(j)},a^{(j)}} u(z)|^2 \Big) \gamma(\complement F_\sigma^*) \le \gamma(\complement F_\sigma^*) \lesssim \gamma(\complement F_\sigma) \le \sum_{k=1}^6 \gamma\big(\{T^*_{(A^{(k)},a^{(k)})} u > \sigma\}\big).
$$

Hence, from (4.8) we infer

$$\gamma_{S_a u}(\sigma) \le \gamma(F_\sigma^* \cap \{S_a u > \sigma\}) + \gamma(\complement F_\sigma^*)$$

$$\lesssim \frac{1}{\sigma^2} \int_{F_\sigma^*} |S_a u(x)|^2 \, d\gamma(x) + \gamma(\complement F_\sigma)$$

$$\lesssim \sum_{j=1}^{6} \left[\gamma_{T^*_{(A^{(j)}, a^{(j)})} u}(\sigma) + \frac{1}{\sigma^2} \int_{F_\sigma} |T^*_{(A^{(j)}, a^{(j)})} u(z)|^2 \, d\gamma(z) \right]$$

$$\lesssim \sum_{j=1}^{6} \left[\gamma_{T^*_{(A^{(j)}, a^{(j)})} u}(\sigma) + \frac{1}{\sigma^2} \int_0^\sigma t \gamma_{T^*_{(A^{(j)}, a^{(j)})} u}(t) \, dt \right].$$

Integrating over σ and noting that

$$\int_0^\infty \frac{1}{\sigma^2} \int_0^\sigma t \gamma_{T^*_{(A^{(j)}, a^{(j)})} u}(t) \, dt \, d\sigma = \int_0^\infty t \gamma_{T^*_{(A^{(j)}, a^{(j)})} u}(t) \int_t^\infty \frac{1}{\sigma^2} \, d\sigma \, dt$$

$$= \int_0^\infty \gamma_{T^*_{(A^{(j)}, a^{(j)})} u}(t) \, dt = \left\| T^*_{(A^{(j)}, a^{(j)})} u \right\|_{L^1(\gamma)},$$

we get, by Theorem 3.1 and with $a^{(j)\prime}$ as in the statement of that theorem,

$$\|S_a u\|_{L^1(\gamma)} \lesssim \sum_{j=1}^{6} \left\| T^*_{(A^{(j)}, a^{(j)})} u \right\|_{L^1(\gamma)} \lesssim \sum_{j=1}^{6} \left\| T^*_{(1, a^{(j)\prime})} u \right\|_{L^1(\gamma)} \le 6 \left\| T^*_{(1, a')} u \right\|_{L^1(\gamma)},$$

where $a' = \max\limits_{j=1,\dots,6} a^{(j)\prime}$.   □

Remark 4.2. In [8] the cones

$$\widetilde{\Gamma}_x^{(A,a)}(\gamma) := \left\{ (y,t) \in \mathbb{R}^n \times (0, \infty) \colon |y - x| < At \text{ and } t < am(y) \right\}$$

are used implicitly. In view of the inclusions

$$\Gamma_x^{(A,a)}(\gamma) \subseteq \widetilde{\Gamma}_x^{(A, c_{a,A})}(\gamma), \qquad \widetilde{\Gamma}_x^{(A,a)}(\gamma) \subseteq \Gamma_x^{(A, c_{a,A})}(\gamma),$$

the corresponding functions $\widetilde{S}$ and $\widetilde{T}^*$ satisfy the pointwise bounds

$$\widetilde{S}_a u(x) \lesssim S_{c_{a,A}} u(x), \qquad S_a u(x) \lesssim \widetilde{S}_{c_{a,A}} u(x)$$

and

$$\widetilde{T}^*_{(A,a)} u(x) \lesssim T^*_{(A, c_{a,A})} u(x), \qquad T^*_{(A,a)} u(x) \lesssim \widetilde{T}^*_{(A, c_{a,A})} u(x).$$

In particular, Theorem 1.1 remains valid if we replace S and T^* by $\widetilde{S}$ and $\widetilde{T}^*$. Remark also that [8, Theorem 3.8] gives a change of aperture formula for tent spaces that implies an analogue of Theorem 3.1 for the square function $\widetilde{S}$ for $A, A' > 1$.

Remark 4.3. We conclude with a few words on reverse inequalities, i.e., controls of the maximal function by the square function. In the euclidean case, such inequalities are generally proven via atomic decompositions, usually going through tent spaces. We have developed, in [8], the gaussian analogues of these spaces and their atomic decomposition. However, to deduce a reverse inequality, we would then need an adequate analogue of the Calderón reproducing formula (analogues exist, but do not seem to be appropriate), and such a formula is involving all $t \in (0, \infty)$ rather than just $t \in (0, am(x))$. A complete Hardy space theory is thus likely to require an understanding of the "non-admissible" parts of objects such as $T^* u$, Su, or Mauceri-Meda's atoms (i.e. the part corresponding to the scales $t \in (am(x), \infty)$,

for which balls are not admissible), or a technique that allows one to avoid such non-admissible part in arguments involving Calderón reproducing formulae. This is the subject of some of our on-going investigations.

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